Tests for Convergence Summary

- Test for Divergence: If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

- The Integral Test: Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent $\Leftrightarrow \int_1^{\infty} f(x)\, dx$ is convergent.

- The Comparison Test: Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.
  (i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.
  (ii) If $\sum b_n$ is divergent and $b_n \leq a_n$ for all n, then $\sum a_n$ is also divergent.

- The Limit Comparison Test: Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where c is a positive, finite number, then either both series converge or both series diverge.
  If the limit is zero and $\sum b_n$ is convergent, then so is $\sum a_n$.
  If the limit is ∞ and $\sum b_n$ is divergent, then so is $\sum a_n$.
- Two special cases:
  - If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ AND if $\sum b_n$ converges, then $\sum a_n$ also converges.
  - If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ AND if $\sum b_n$ diverges, then $\sum a_n$ also diverges.

- The Alternating Series Test: If the alternating series
  $\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$, where $b_n > 0$, satisfies
  (i) $b_{n+1} \leq b_n$ for all n
  (ii) $\lim_{n\to\infty} b_n = 0$
  then the series is convergent.

- The Ratio Test:
  (i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum a_n$ is convergent
  (ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or is ∞, then the series $\sum a_n$ is divergent
  (iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive

- The Root Test:
  (i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum a_n$ is convergent
  (ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or is ∞, then the series $\sum a_n$ is divergent
  (iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive

- The remainder of a partial sum and estimating sums: Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent with sum s. Then if S_n is a partial sum,
  - $R_n = s - s_n = a_{n+1} + a_{n+2} + \cdots$ is the remainder in approximating s
  - $\int_{n+1}^{\infty} f(x)\, dx \leq R_n \leq \int_n^{\infty} f(x)\, dx$
  - $s_n + \int_{n+1}^{\infty} f(x)\, dx \leq s \leq s_n + \int_n^{\infty} f(x)\, dx$

  (continued)

Tests for Convergence Summary

- The Alternating Series Estimation Theorem: If $S = \sum_{n=1}^{\infty}(-1)^{n-1}b_n$ is a series that converges by the Alternating Series Test, then $|R_n| \leq b_{n+1}$.